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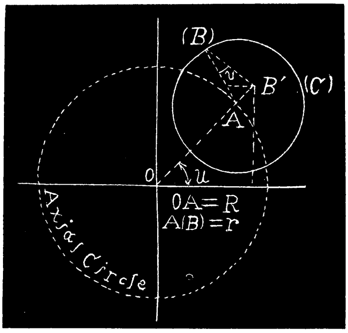
NOTE ON THE LOXODROMIC LINES OF THE TORUS.*

By Dr. ARNOLD EMCH, Professor of Graphic Mathematics, Kansas State Agricultural College, Manhattan, Kas.

1. Designating by u and v the angles, which, in Fig. 1, determine the position of a point B on the surface of a torus, the square of a linear element on the surface has the form,

$$ds^2=(R+r\sin v)^2(du_1^2+dv_1^2) \quad \ldots\ldots(1),$$

$$\text{where } u_1=u \text{ and } v_1=\int_0^v \frac{dv}{\frac{R}{r}+\sin v} \quad \ldots\ldots(2).$$



(C)=Revolved Meridian.

Fig. 1.

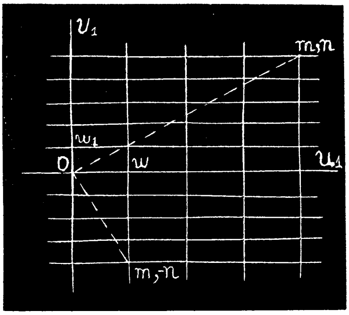


Fig. 2.

R and r are respectively the radii of the axial circle and of a meridian of the torus. As it is well known, by means of the expressions (2), the points of the torus are conformally transformed into the points of a rectangle. Considering exclusively the case where $R>r$ the integral v_1 has the value

$$v_1=\frac{r}{\sqrt{R^2-r^2}}\text{arc sin}\left(\frac{1+\frac{R}{r}\sin v}{\frac{R}{r}+\sin v}\right) \quad \ldots\ldots(3).$$

The values of u and v can each vary from 0 to 2π. In this interval u_1 varies from 0 to 2π and according to the expression (3), v_1 from

$$\frac{r}{\sqrt{R^2-r^2}}\text{arc sin}\frac{r}{R} \text{ to } \frac{r}{\sqrt{R^2-r^2}}\left[\text{arc sin}\frac{R}{r}+2\pi\right].$$

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*A preliminary statement concerning orthographic loxodromics on a torus was made by the author in a paper, "Ueber orthogonale Systeme und einige technische Anwendungen," which appeared in the program of the Polytechnic of Biel, 1898.

The sides of the primitive rectangle are therefore

$$w=2\pi \text{ and } w_1=\frac{r2\pi}{\sqrt{R^2-r^2}} \ldots\ldots(4).$$

To the lines $u_1=\text{const.}$, $v_1=\text{const.}$, respectively, parallel to the sides of the rectangle (w, w_1), correspond on the torus the meridians and parallels, and vice versa to the meridians and parallels $u=\text{const.}$ and $v=\text{const.}$, correspond in the rectangle lines, respectively, parallel to the sides w and w_1.

2. Evidently these lines form isothermal systems. If we now put $u_1+iv_1=z$, and if α and β designate two complex quantities $\alpha=p-iq$, $\beta=r(1+i)$, the monogenic function

$$\varphi+i\psi=\alpha z+\beta$$

represents also an isothermal system in the (u_1, v_1)-plane. There is

$$\varphi=au_1+bv_1+c \ldots\ldots(5),$$

$$\psi=-bu_1+av_1+c \ldots\ldots(6),$$

which represent two perpendicular pencils of parallel rays. Interpreted by means of formulae (2) they represent two systems of orthographic loxodromic lines on the torus.

3. Among the loxodromic lines of the torus we shall consider those that close after a certain number of revolutions around the axis and the axial circle of the torus. For this purpose it will be well to refer to the integral

$$Z=\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \ldots\ldots(7),$$

by which the positive part of the z-plane is transformed into the rectangle of the z-plane whose sides correspond to the periods of the elliptic function

$$z=\lambda(Z) \ldots\ .(8),$$

as defined by the elliptic integral (7).[*]

In order that the sides of the rectangle assume the values w and w_1, the modulus k of integral (7) must be chosen in such a manner that according to a well-known formula

$$k=4\sqrt{q}\left[\frac{(1+q^2)(1+q^4)(1+q^6)\ldots.}{(1+q)(1+q^3)(1+q^5)\ldots..}\right]^4,$$

where $q=e^{-(2\pi w_1)/w}$.

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[*] See F. Klein, Ueber Riemann's Theorie der Algebraische Funktionen und ihrer Integrale, pages 50—55.

Supposing that $z=\lambda(Z)$ has been determined in this way, it is possible to transform the surface of the torus conformally upon the positive part of the z-plane and to express the periodic character of this transformation. The general period of $z=\lambda(Z)$ has the form

$$W=mw+niw,$$

or also

$$W=2\pi\left(m+ni\,\frac{r}{\sqrt{R^2-r^2}}\right)\ldots\ldots(9).$$

By means of this expression for the period it is easy to show an interesting relation which exists between two orthographic loxodromic lines of a torus. From the origin O of the system of parallelograms of periods draw two lines, one to the point $\left(2m\pi,\ 2ni\pi\frac{r}{\sqrt{R^2-r^2}}\right)$; other to point $\left(2m_1\pi,\ -2n_1i\pi\frac{r}{\sqrt{R^2-r^2}}\right)$ of the system, (Fig. 2). The trigonometric tangents of these lines with the positive part of the u_1-axis are

$$u=\frac{n}{m}.\frac{r}{\sqrt{R^2-r^2}},\quad v=-\frac{n_1}{m_1}.\frac{r}{\sqrt{R^2-r^2}},$$

and the tangent of the angle included by the two lines is

$$\frac{\frac{n}{m}.\frac{r}{\sqrt{R^2-r^2}}+\frac{n_1}{m_1}.\frac{r}{\sqrt{R^2-r^2}}}{1-\frac{n}{m}.\frac{n_1}{m_1}.\frac{r}{\sqrt{R^2-r^2}}}.$$

The condition for a right angle is

$$\frac{n}{m}.\frac{n_1}{m_1}=\frac{r}{\sqrt{R^2-r^2}}\ldots\ldots(10).$$

In this case the corresponding loxodromic lines on the torus are orthographic and since in the plane $(u_1,\ v_1)$ the lines end in points of periods, it is clear that the two orthographic loxodromic lines close. If the numbers m and n are given, and provided R/r is a rational fraction, then it is always possible to find two integers m_1 and n_1 so that relation (10) is satisfied. Evidently m gives the number of revolutions around the axis (perpendicular to the parallels) and n the number of revolutions around the axial circle of the torus. Thus we are led to the theorem :

If the ratio R/r of a torus is a rational fraction and if a loxodromic line of the torus closes after m revolutions around the axis and n revolutions around the *axial circle of the torus, then every orthographic loxodromic line closes after* m_1 *and* n_1 *revolutions around the same axes, so that*

$$\frac{n}{m}\cdot\frac{n_1}{m_1}=\frac{r}{\sqrt{R^2-r^2}}.$$

4. Among the great number of special cases, which arise from this relation, we shall simply mention the case where $m=n=1$. The following theorem is easily found:

The loxodromic lines of a torus, which turn once around the axis and once around the axial circle of the torus, are the circles formed by the intersection of the double-tangent planes with the torus.

The results found about loxodromic lines of a torus may immediately be generalized for those cyclides which arise from the torus by circular transformations.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

108. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A man who feels his death approach bequeaths to his young wife one-third of his fortune, and the remaining two-thirds to his son, if such should be born; but one-half of it to the widow and the other half to his daughter, if such should be born. After his death, twins are born, a son and a daughter. How should the fortune be divided amongst the three?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.; P. S. BERG, B. S., Principal of Schools, Larimore, N. D.; SYLVESTER ROBINS, North Branch, N. J.; JOSIAH H. DRUMMOND, LL. D., Portland, Me.; ELMER SCHUYLER, High Bridge, N. J.; and ALOIS F. KOVARIK, Professor of Mathematics and Physics, Decorah Institute, Decorah, Ia.

By the conditions of the problem, the wife is to have as much as the daughter and half as much as the son.

$\therefore$ son : wife : daughter$=2:1:1$.

$\therefore$ son should receive $\frac{1}{2}$ the fortune, wife should receive $\frac{1}{4}$ the fortune, and daughter should receive $\frac{1}{4}$ the fortune.

II. Solution by D. G. DORRANCE, Camden, N. Y.

Consider the fortune (a) divided into two fortunes each of one-half the value of the whole, viz: $\frac{1}{2}a$ and $\frac{1}{2}a$.